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On Spectral Radius of Biased Random Walks on Infinite Graphs*

He Song† Zhan Shi‡ Vladas Sidoravicius§ Longmin Wang¶
Kainan Xiang‖

Abstract

We consider a class of biased random walks on infinite graphs and present several general results on the spectral radius of biased random walk.

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Let $G = (V(G), E(G))$ be a locally finite, connected infinite graph, where $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. Fix a vertex o of G as the root, we assume that o has at least one neighbor. For any reversible Markov chain on G, there is a stationary measure $\pi(\cdot)$ such that for any two adjacent vertices x and y, $\pi(x)p(x, y) = \pi(y)p(y, x)$, where $p(x, y)$ is the transition probability of the Markov chain. For the edge joining vertices x and y, we assign a weight

$$c(x,y) = \pi(x)p(x, y),$$

and call by *conductances* the weights of the edges. We study the biased random walks on the rooted graph (G, o) defined as follows:
For any vertex x of G let $|x|$ denote the graph distance between x and o. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{Z}_+$:

$$B_G(n) = \{x \in V(G) : \ |x| \le n\}, \qquad \partial B_G(n) = \{x \in V(G) : \ |x| = n\}.$$

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†Huaiyin Normal UniversityčňP. R. China E-mail: songhe@hytc.edu.cn
‡LPSM, Sorbonne Université Paris VIčňFrance E-mail: zhan.shi@upmc.fr
§NYU Shanghai & Courant Institute of Mathematical SciencesčňP. R. China E-mail: vs1138@nyu.edu
¶Nankai University, P. R. China. E-mail: wanglm@nankai.edu.cn
‖Xiangtan UniversityčňP. R. China E-mail: kainan.xiang@xtu.edu.cn

Let $\lambda \in [0, \infty)$. For $\lambda > 0$, if an edge $e = \{x, y\}$ is at distance n from o, i.e., $\min(|x|, |y|) = n$, its conductance is defined as λ^{-n}. Denote by RW_λ the nearest-neighbour random walk $(X_n)_{n=0}^{\infty}$ among such conductances and call it the λ-*biased random walk*. In other words, RW_λ has the following transition probabilities: for $v \sim u$ (i.e., if u and v are adjacent on G),

$$p(v,u) := p_\lambda^G(v,u) = \begin{cases} \frac{1}{d_v} & \text{if } v = o, \\ \frac{\lambda}{d_v + (\lambda-1)d_v^-} & \text{if } u \in \partial B_G(|v|-1) \text{ and } v \neq o, \\ \frac{1}{d_v + (\lambda-1)d_v^-} & \text{otherwise.} \end{cases} \tag{0.1}$$

Here, d_v is the degree of vertex v, and d_v^-, d_v^0 and d_v^+ are the numbers of edges connecting v to $\partial B_G(|v|-1)$, $\partial B_G(|v|)$ and $\partial B_G(|v|+1)$ respectively. Note that

$$d_v^+ + d_v^0 + d_v^- = d_v, \qquad d_v^- \geq 1, \qquad v \neq o, \qquad d_o^- = d_o^0 = 0,$$

and that $\mathrm{RW}_{\lambda=1}$ is the simple random walk (SRW) on G. When $\lambda = 0$, for $v \sim u$ define

$$p(v,u) := p_\lambda^G(v,u) = \begin{cases} \frac{1}{d_v} & \text{if } v = o, \\ \frac{1}{d_v^-} & \text{if } u \in \partial B_G(|v|-1) \text{ and } v \neq o \ d_v^+ = d_v^0 = 0, \\ \frac{1}{d_v - d_v^-} & \text{otherwise.} \end{cases} \tag{0.2}$$

By Rayleigh's monotonicity principle (see [28], p. 35), there is a critical value $\lambda_c(G) \in (0, \infty]$ such that RW_λ is transient for $\lambda < \lambda_c(G)$ and is recurrent for $\lambda > \lambda_c(G)$. Let $M_n = \#(\partial B_G(n))$ be the cardinality of $\partial B_G(n)$ for any $n \in \mathbb{Z}_+$. Define the volume growth rate of G as

$$\mathrm{gr}(G) = \liminf_{n\to\infty} M_n^{1/n}.$$

When G is a tree, $\lambda_c(G)$ is exactly the exponential of the Hausdorff dimension of the tree boundary, namely the branching number of the tree ([14], [22], [28]). When G is a transitive graph, $\lambda_c(G) = \mathrm{gr}(G)$ (see [24] and [28]). Let

$$\mathrm{gr}_+(G) = \liminf_{n\to\infty} \Big(\sum_{x \in \partial B_G(n-1)} d_x^+ \Big)^{1/n}.$$

Clearly $\mathrm{gr}_+(G) \geq \mathrm{gr}(G)$. If G either is a tree or satisfies

$$\limsup_{n\to\infty} \left(\max_{|x|=n} d_x^+ \right)^{1/n} = 1,$$

then $\mathrm{gr}_+(G) = \mathrm{gr}(G)$.

From the Nash-Williams criterion ([28] Section 2.5), it follows that for any G with $\mathrm{gr}_+(G) < \infty$, RW_λ is recurrent for $\lambda > \mathrm{gr}_+(G)$ and thus $\lambda_c(G) \leq \mathrm{gr}_+(G)$. If G is spherically symmetric then $\lambda_c(G) = \mathrm{gr}_+(G)$ ([28] Section 3.4, Exercise 3.11).

An original motivation for introducing RW_λ by Berretti and Sokal [8] was to design a Monte-Carlo algorithm for self-avoiding walks. See [20, 32, 29] for refinements of this idea. Since the 1980s biased random walks and biased diffusions in disordered media have attracted much attention in mathematical and physics communities due to their interesting phenomenology and similarities to concrete physical systems ([3, 10, 11, 17]). In the 1990s, Lyons ([22, 23, 24]), and Lyons, Pemantle and Peres ([25, 26]) made a fundamental advance in the study of RW_λ's. RW_λ has also received attention recently, see [6, 2, 5, 18] and the references therein. For a survey on biased random walks on random graphs see Ben Arous and Fribergh [4].

This paper focuses on a specific properties of spectral radius of RW_λ's on non-random infinite graphs. The uniform spanning forests of the network associated with RW_λ on the Euclidean lattices are studied in a companion paper [30]. Recently, the continuity and analyticity of random walk quantities such as the rate of escape and the entropy rate as functions of the transition probabilities have been studied by various authors. See for example Erschler [12], Ledrappier [21], Erschler and Kaimanovich [13], Gilch and Ledrappier [15], Gouëzel [16] for corresponding results of simple random walks on hyperbolic groups and general finitely generated groups.

Let us introduce some basic notation. Write

$$p^{(n)}(x,\, y) := p_\lambda^{(n)}(x,\, y) = \mathbb{P}_x(X_n = y),$$

where $\mathbb{P}_x := \mathbb{P}_x^G$ is the law of RW_λ starting at x. The Green function is given by

$$\mathbb{G}(x,\, y \,|\, z) := \mathbb{G}_\lambda(x,\, y \,|\, z) = \sum_{n=0}^{\infty} p^{(n)}(x,\, y) z^n,\ x,\ y \in V(G),\ z \in \mathbb{C},\ |z| < R_{\mathbb{G}}\,,$$

where $R_{\mathbb{G}} = R_{\mathbb{G}}(\lambda) = R_{\mathbb{G}}(\lambda, x, y)$ is its convergence radius. Note that

$$R_{\mathbb{G}} = R_{\mathbb{G}}(\lambda) = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}}$$

is independent of x, y when RW_λ is irreducible, i.e., $\lambda > 0$. Call

$$\rho_\lambda = \rho(\lambda) = \frac{1}{R_{\mathbb{G}}} = \limsup_{n\to\infty} p^{(n)}(x,x)^{1/n} = \limsup_{n\to\infty} p^{(n)}(o,o)^{1/n}$$

the spectral radius of RW_λ.

We are ready to state our main results. The proofs will be presented in Section 1 and 2.

Theorem 0.1. *Let G be a locally finite, connected infinite graph.*

(i) *The spectral radius ρ_λ is continuous in $\lambda \in (0,\infty)$, and $\rho(\lambda_c) = 1$.*

(ii) *If the limit of ρ_λ continuous at 0, then there are no adjacent vertices in $\partial B_G(n)$ for any $n \in \mathbb{N}$, and $d_v - d_v^- \geq 1$ for any vertex v.*

Conversely, on any infinite graph G, if for any $n \in \mathbb{N}$ there are no adjacent vertices in $\partial B_G(n)$, and if there exists $\delta > 0$ such that $d_v - d_v^- \geq \delta d_v$ for any vertex v, then ρ_λ is continuous at 0.

Let $d \in \mathbb{N}$, $d \geq 2$, and $\mathcal{G}_d$ denotes the set of all d-regular infinite connected graphs, $\mathbb{T}_d$ denotes d-regular trees.

Theorem 0.2. *Let $G \in \mathcal{G}_d$, and $\lambda \in (0,\, \lambda_c(\mathbb{T}_d) = d-1)$.*

(i) *We have*

$$\rho_G(\lambda) \geq \rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}\,.$$

(ii) *Assume G is transitive. Then*

$$\rho_G(\lambda) = \rho_{\mathbb{T}_d}(\lambda) \text{ if and only if } G \text{ is isomorphic to } \mathbb{T}_d.$$

In the case $\lambda = 1$, Theorem 0.2 follows from Kesten [19, Theorem 2] (see also [33, p. 122 Corollary 11.7] and [28, Theorem 6.11]). Furthermore, Abért, Glasner and Virág [1] provided a quantitative strengthening of Kesten's theorem and extended it to unimodular random rooted regular graphs. See also Lyons and Peres [27] for relation between the equality $\rho_G(1) = \rho_{\mathbb{T}^d}(1)$ and the frequency of times spent by simple random walk in a nontrivial cycle on general regular graphs.

When emphasizing that a function $g(\cdot)$ depends on the underlying graph G, we will use $g_G(\cdot)$ or $g^G(\cdot)$ to replace $g(\cdot)$.

1 Proofs of Theorem 0.1

For any vertex set A, let

$$\tau_A = \inf\{n \geq 0 \,|\, X_n \in A\}, \qquad \tau_A^+ = \inf\{n \geq 1 \,|\, X_n \in A\}.$$

When $A = \{y\}$, write $\tau_y = \tau_{\{y\}}$, $\tau_y^+ = \tau_{\{y\}}^+$. Put

$$f^{(n)}(x,\, y) := f_\lambda^{(n)}(x,\, y) = \mathbb{P}_x(\tau_y^+ = n), \tag{1.1}$$

$$U(x,\, y \,|\, z) := U(x,\, y \,|\, z) = \sum_{n=1}^{\infty} f^{(n)}(x,\, y) z^n,\ x,\ y \in V(G), \qquad z \in \mathbb{C},\ |z| < R_U \tag{1.2}$$

where $R_U = R_U(\lambda) = R_U(\lambda,\, x,\, y)$ is the convergence radius of U, which is also independent of x, y for $\lambda > 0$. When $\lambda = 0$, $R_U(0) = \infty$.

1.1 Proof of Theorem 0.1 part (i)

Proof. It suffices to verify that the convergence radius $R_{\mathbb{G}}(\lambda)$ is continuous in $\lambda \in (0,\, \infty)$. This is done in two steps.

Step 1. For any sequence $\{\lambda_k\}_{k\geq 1} \subset (0,\, \lambda_c(G)]$ converging to a limit $\lambda_0 \in (0,\, \lambda_c(G)]$, we claim that

$$\limsup_{k\to\infty} R_{\mathbb{G}}(\lambda_k) \leq R_{\mathbb{G}}(\lambda_0) \leq \liminf_{k\to\infty} R_{\mathbb{G}}(\lambda_k).$$

For any $n \in \mathbb{Z}_+$, let

$$\begin{aligned} \Pi_n &= \{\text{paths } \gamma \text{ in } G \text{ staring and ending at } o \text{ with length } n\}, \\ \mathbb{P}(\gamma,\, \lambda) &= \prod_{i=0}^{n-1} p_\lambda(w_i, w_{i+1}),\ \gamma = w_0 w_1 \cdots w_n \in \Pi_n \,. \end{aligned}$$

Note that for $0 < \lambda_1 \leq \lambda_2 < \infty$ and $v \sim u$ we have

$$\frac{\lambda_1}{\lambda_2} \leq \frac{p_{\lambda_1}(v,\, u)}{p_{\lambda_2}(v,\, u)} \leq \frac{\lambda_2}{\lambda_1} \,. \tag{1.3}$$

Thus, for any $1 > \delta > 0$, there is a constant $\varepsilon > 0$ such that $(1-\delta)p_{\lambda_0}(v,\, u) \leq p_\lambda(v,\, u) \leq (1+\delta)p_{\lambda_0}(v,\, u)$ for $\lambda \in (\lambda_0 - \varepsilon,\, \lambda_0 + \varepsilon)$. Consequently, we have $(1-\delta)^n \mathbb{P}(\gamma,\, \lambda_0) \leq \mathbb{P}(\gamma,\, \lambda) \leq (1+\delta)^n \mathbb{P}(\gamma,\, \lambda_0)$ for $\gamma \in \Pi_n$ and

$$p_\lambda^{(n)}(o,\, o) = \sum_{\gamma\in\Pi_n} \mathbb{P}(\gamma,\, \lambda) \geq \sum_{\gamma\in\Pi_n} (1-\delta)^n \mathbb{P}(\gamma,\, \lambda_0) = (1-\delta)^n p_{\lambda_0}^{(n)}(o,\, o),$$

$$p_\lambda^{(n)}(o,\, o) = \sum_{\gamma\in\Pi_n} \mathbb{P}(\gamma,\, \lambda) \leq \sum_{\gamma\in\Pi_n} (1+\delta)^n \mathbb{P}(\gamma,\, \lambda_0) = (1+\delta)^n p_{\lambda_0}^{(n)}(o,\, o).$$

Therefore we have for k large enough,

$$\mathbb{G}_{\lambda_k}(o,\, o \,|\, z) = \sum_{n=0}^{\infty} p_{\lambda_k}^{(n)}(o,\, o) z^n \geq \sum_{n=0}^{\infty} p_{\lambda_0}^{(n)}(o,\, o)\, ((1-\delta)z)^n > -\infty,$$

$$\mathbb{G}_{\lambda_k}(o,\, o\,|\, z) = \sum_{n=0}^{\infty} p_{\lambda_k}^{(n)}(o,\, o) z^n \le \sum_{n=0}^{\infty} p_{\lambda_0}^{(n)}(o,\, o)\, ((1+\delta)z)^n < \infty,$$

provided $(1+\delta)z < R_{\mathbb{G}}(\lambda_0)$. Since δ is arbitrary, we have that $\liminf_{k\to\infty} R_{\mathbb{G}}(\lambda_k) \ge R_{\mathbb{G}}(\lambda_0) \ge \limsup_{k\to\infty} R_{\mathbb{G}}(\lambda_k)$.

Step 2. It remains to prove $R_{\mathbb{G}}(\lambda_c) = 1$. Suppose $R_{\mathbb{G}}(\lambda_c) > 1$, then for $\lambda > \lambda_c$ and $z > 1$ with $1 < \frac{\lambda z}{\lambda_c} < R_{\mathbb{G}}(\lambda_c)$, we would have from (1.3) that

$$\sum_{n=0}^{\infty} p_{\lambda}^{(n)}(o,\, o) z^n \le \sum_{n=0}^{\infty} p_{\lambda_c}^{(n)}(o,\, o) \left(\frac{\lambda z}{\lambda_c}\right)^n < \infty.$$

Then $R_{\mathbb{G}}(\lambda) > 1$. This contradicts to the fact that RW_λ is recurrent for $\lambda > \lambda_c$.

1.2 Proof of Theorem 0.1 part (ii)

We split the proof of (ii) into three steps.

Step 1. For any given locally finite, connected infinite graph G, such that $\partial B_G(n_0)$ contains adjacent vertices for some n_0 we prove that ρ_λ is not continuous at 0.

Let u and v be adjacent vertices in $\partial B_G(n_0)$. Let $e = \{u,\, v\}$. For RW_λ (with $\lambda > 0$, starting at u) to return to u, it suffices to walk $2n$ steps between u and v. Accordingly,

$$p_{\lambda}^{(2n)}(u,\, u) \ge \left(\frac{1}{d_u + (\lambda-1)d_u^-}\right)^n \left(\frac{1}{d_v + (\lambda-1)d_v^-}\right)^n. \tag{1.4}$$

So for any $\lambda > 0$,

$$\rho_\lambda \ge \limsup_{n\to\infty} \left\{p_{\lambda}^{(2n)}(u,\, u)\right\}^{\frac{1}{2n}} \ge \frac{1}{\{[d_u + (\lambda-1)d_u^-]\,[d_v + (\lambda-1)d_v^-]\}^{1/2}} > 0.$$

Letting $0 < \lambda \to 0$, we immediately get

$$\liminf_{\lambda\to 0+} \rho_\lambda \ge \frac{1}{[(d_u - d_u^-)(d_v - d_v^-)]^{1/2}} > 0 = \rho_0.$$

Step 2. Assume that there is a vertex v such that $d_v - d_v^- = 0$. Similar to the arguments in the previous step, we have for any n,

$$p_{\lambda}^{(2n)}(u,\, u) \ge \Big(\frac{1}{d_u + (\lambda-1)d_u^-}\Big)^n \Big(\frac{1}{d_v}\Big)^n.$$

Then for any $\lambda > 0$,

$$\rho_\lambda \ge \Big(\frac{1}{d_v(d_u + (\lambda-1)d_u^-)}\Big)^{1/2} > 0.$$

Hence ρ_λ is not continuous at 0.

Step 3. Assume that there are no adjacent vertices in $\partial B_G(n)$ for any $n \in \mathbb{N}$, and there exists $\delta > 0$ such that $d_v - d_v^- \ge \delta d_v$ for any vertex v. Then for any $\lambda > 0$ and the RW_λ $(X_n)_{n=0}^\infty$, the following holds almost surely:

$$|X_{n+1}| - |X_n| \in \{+1, -1\}, \qquad \forall n \in \mathbb{Z}_+. \tag{1.5}$$

When $X_0 = o$, the walk $(X_n)_{n=0}^\infty$ takes an even number (say, $2m$, for some $m \ge 1$) of steps to return to o: Among these $2m$ steps, m steps are upward and the other m steps are downward. When $v \sim u$ and $|u| = |v| - 1$, we have

$$p_\lambda(v,\, u) = \frac{\lambda}{d_v + (\lambda-1)d_v^-} \le \frac{\lambda}{d_v - d_v^-} \le \lambda d_v^{-1}\delta^{-1}, \qquad \lambda > 0.$$

When $v \sim u$ and $|u| = |v| + 1$, we have $p_\lambda(v,\, u) \le d_v^{-1}\delta^{-1}$. Hence for any path $\gamma = w_0 w_1 \cdots w_{2n} \in \Pi_{2n}$,

$$\mathbb{P}(\gamma,\, \lambda) = \prod_{i=0}^{2n-1} p_\lambda(w_i,\, w_{i+1}) \le \lambda^n \delta^{-2n} \mathbb{P}(\gamma, 1), \qquad \lambda > 0,$$

which implies that for any $\lambda > 0$,

$$p_\lambda^{(2n)}(o,\, o) = \sum_{\gamma \in \Pi_{2n}} \mathbb{P}(\gamma,\, \lambda) \le \lambda^n \delta^{-2n} \sum_{\gamma \in \Pi_{2n}} \mathbb{P}(\gamma, 1) \le \lambda^n \delta^{-2n} p_1^{(2n)}(o, o).$$

Hence

$$\rho_\lambda = \limsup_{n\to\infty} \left\{ p_\lambda^{(2n)}(o,\, o) \right\}^{\frac{1}{2n}} \le \delta^{-1} \rho_1 \lambda^{1/2},$$

proving that $\lim_{\lambda\to 0+} \rho_\lambda = 0 = \rho_0$.

□

2 Proof of Theorem 0.2

We start with the lemma, which will be used in the proof of Theorem 0.2.

Lemma 2.1. *[31] For the d-regular tree $\mathbb{T}_d$, the following holds:*

$$\rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}, \qquad \lambda \in [0,\, \lambda_c(\mathbb{T}_d)] = [0,\, d-1],$$

and for $\lambda \in (0,\, \infty)$ and $n \to \infty$,

$$f_\lambda^{(2n)}(o,\, o) \sim \frac{1}{\sqrt{\pi}} \left(\frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda} \right)^{2n} n^{-3/2}. \tag{2.1}$$

Moreover,

$$p_\lambda^{(2n)}(o, o) \sim \begin{cases} \frac{(d-1-\lambda)^2}{16(\pi\lambda)^{1/2}(d-1)^{3/2}} \rho_{\mathbb{T}_d}(\lambda)^{2n} n^{-3/2} & \text{if } \lambda \in (0,\, d-1), \\ \frac{1}{\sqrt{\pi n}} & \text{if } \lambda = d-1. \end{cases} \tag{2.2}$$

Now we are ready to give the proof of Theorem 0.2.

Proof of Theorem 0.2. (i) Fix $\lambda \in (0,\, \lambda_c(\mathbb{T}_d))$. Define $g = g_\lambda :\ \mathbb{Z}_+ \to \mathbb{R}$ by

$$g(n) = g_\lambda(n) := \left(1 + \frac{d-1-\lambda}{d-1+\lambda} n \right) ((d-1)/\lambda)^{-n/2},$$

and $f = f_\lambda :\ G \to \mathbb{R}$ by

$$f(x) := f_\lambda(x) = g(|x|), \qquad \forall x \in V(G). \tag{2.3}$$

It is easy to see that $g(0) = 1$ and $g(1) = \left(\frac{2(d-1)}{d-1+\lambda}\right)((d-1)/\lambda)^{-1/2}$. Set $a = \frac{\lambda}{d-1}$, hence $0 < a < 1$ and $g(1) = \frac{2a^{1/2}}{1+a} < 1 = g(0)$. For $n \in \mathbb{Z}$,

$$\begin{aligned} \frac{g(n+1)}{g(n)} &= \frac{(1 + \frac{1-a}{1+a}(n+1))a^{\frac{n+1}{2}}}{(1 + \frac{1-a}{1+a}n)a^{\frac{n}{2}}} \\ &= \frac{n+2-na}{n+1-(n-1)a} a^{\frac{1}{2}} \\ &= (1 + \frac{1-a}{2+(n-1)(1-a)}) a^{\frac{1}{2}}. \end{aligned}$$

It is easy to see that $\frac{1-a}{2+(n-1)(1-a)}$ is strictly decreasing on $\mathbb{Z}$. So $\frac{g(n+1)}{g(n)}$ obtains the maximum value $\frac{(3-a)a^{1/2}}{2}$ when $n=1$. For Consider function $k(a)=\left(\frac{(3-a)a^{1/2}}{2}\right)^2=\frac{(3-a)^2a^2}{4}$, $(0<a<1)$, we can check that $k(a)$ in strictly increasing on $0<a<1$ and $k(a)<1$. So g is strictly decreasing on $\mathbb{Z}_+$. Recall $p(x,y)$ from (0.2). For any $h:\ G\to\mathbb{R}$, let

$$Ph(x):=\sum_{y\sim x}p(x,\,y)h(y),\qquad x\in V(G). \tag{2.4}$$

Then $Pf(o)=\rho_{\mathbb{T}_d}(\lambda)f(o)$, and for $x\neq o$,

$$Pf(x)=\frac{d_x^+g(|x|+1)+d_x^0g(|x|)+\lambda d_x^-g(|x|-1)}{d_x^++d_x^0+\lambda d_x^-}$$

$$\geq\frac{(d_x^++d_x^0)g(|x|+1)+\lambda d_x^-g(|x|-1)}{d_x^++d_x^0+\lambda d_x^-}.$$

Since $g(|x|-1)\geq g(|x|+1)$ and $d_x^-\geq 1$ (so $d_x^++d_x^0\leq d-1$), this leads to:

$$Pf(x)\geq\frac{(d-1)g(|x|+1)+\lambda g(|x|-1)}{d-1+\lambda}=\rho_{\mathbb{T}_d}(\lambda)f(x),\qquad x\neq o. \tag{2.5}$$

For further use, we notice that for $x\neq o$, if $Pf(x)=\rho_{\mathbb{T}_d}(\lambda)f(x)$, then $d_x^-=1$, $d_x^0=0$ and $d_x^+=d-1$.

For any $n\in\mathbb{N}$, put $f_n:=f\,I_{B_G(n)}$. For $x\in B_G(n)$,

$$Pf_n(x)=Pf(x)-\frac{d_x^+g(n+1)}{d_x^++d_x^0+\lambda d_x^-}I_{\{|x|=n\}}.$$

Define μ as follows: $\mu(o)=d_o$ and $\mu(x)=(d_x^++d_x^0+\lambda d_x^-)\lambda^{-|x|}$ for $x\neq o$. Let $M_n:=|\partial B_G(n)|$ as before. Denote by $(\,\cdot\,,\,\cdot\,)$ the inner product of $L^2(G,\mu)$. Then

$$(Pf_n,\,f_n)=\sum_{x\in B_G(n)}Pf(x)f(x)\mu(x)-\sum_{x\in\partial B_G(n)}\frac{d_x^+g(n+1)}{d_x^++d_x^0+\lambda d_x^-}f(x)\mu(x)\,.$$

For the sum $\sum_{x\in B_G(n)}$ on the right-hand side, we observe that by (2.5), for $x\in B_G(n)$, $Pf(x)\geq\rho_{\mathbb{T}_d}(\lambda)f(x)=\rho_{\mathbb{T}_d}(\lambda)f_n(x)$. For the sum $\sum_{x\in\partial B_G(n)}$, we note that for $x\in\partial B_G(n)$, since $d_x^+\leq d-1$ and $f(x)=g(n)$, we have $\frac{\mu(x)}{d_x^++d_x^0+\lambda d_x^-}=\lambda^{-n}$. Accordingly,

$$(Pf_n,\,f_n)\geq\rho_{\mathbb{T}_d}(\lambda)(f_n,\,f_n)-(d-1)M_n\,g(n)g(n+1)\,\lambda^{-n}\quad\geq\rho_{\mathbb{T}_d}(\lambda)(f_n,\,f_n)-(d-1)M_n\,g(n)^2\lambda^{-n},$$

which implies that

$$\rho_G(\lambda)=\sup_{h\in L^2(G,\mu)\setminus\{0\}}\frac{(Ph,\,h)}{(h,\,h)}\geq\frac{(Pf_n,\,f_n)}{(f_n,\,f_n)}\geq\rho_{\mathbb{T}_d}(\lambda)-\frac{(d-1)M_n\,g(n)^2\lambda^{-n}}{(f_n,\,f_n)}\,.$$

Observe that

$$(f_n,\,f_n)=\sum_{k=0}^{n}\sum_{x\in\partial B_G(k)}g(k)^2\mu(x)=\sum_{k=0}^{n}\sum_{x\in\partial B_G(k)}g(k)^2\,(d_x^++d_x^0+\lambda d_x^-)\,\lambda^{-|x|}$$

$$\geq(\lambda\wedge 1)d\sum_{k=0}^{n}M_k\,g(k)^2\,\lambda^{-k}.$$

Hence

$$\rho_G(\lambda) \geq \rho_{\mathbb{T}_d}(\lambda) - \frac{d-1}{d(\lambda \wedge 1)} \frac{M_n\, g(n)^2\, \lambda^{-n}}{\sum_{k=0}^n M_k\, g(k)^2\, \lambda^{-k}}.$$

It remains to prove that

$$\lim_{n\to\infty} \frac{M_n\, g(n)^2\, \lambda^{-n}}{\sum_{k=0}^n M_k\, g(k)^2\, \lambda^{-k}} = 0.$$

For $k \leq n$,

$$M_n\, g(n)^2\, \lambda^{-n} \leq M_k\, (d-1)^{n-k} g(n)^2\, \lambda^{-n} = M_k\, g(k)^2\, \lambda^{-k} \left(\frac{(d-1-\lambda)n + d-1+\lambda}{(d-1-\lambda)k + d-1+\lambda} \right)^2,$$

which implies that

$$\frac{\sum_{k=0}^n M_k\, g(k)^2\, \lambda^{-k}}{M_n\, g(n)^2\, \lambda^{-n}} \geq \sum_{k=0}^n \left(\frac{(d-1-\lambda)k + d-1+\lambda}{(d-1-\lambda)n + d-1+\lambda} \right)^2.$$

Since $\lambda \leq d-1$, the sum on the right-hand side goes to infinity as $n \to \infty$.

(ii) For $d = 2$, $\mathcal{G}_d = \{\mathbb{T}_2\}$, the result holds trivially. So we assume $d \geq 3$. It suffices to prove that for any transitive $G \in \mathcal{G}_d$ with the minimal cycle length $\ell \geq 3$,

$$\rho_G(\lambda) > \rho_{\mathbb{T}_d}(\lambda), \qquad \forall \lambda \in (0,\, \lambda_c(\mathbb{T}_d)). \tag{2.6}$$

Step 1. $\lambda_c(G) < \lambda_c(\mathbb{T}_d) = d-1$.

Let $\Gamma_{d,\ell} := \langle a_1, \ldots, a_{d-2},\, b \,|\, a_i^2 = 1,\ b^\ell = 1 \rangle$ be a finitely-presented group with generating set $S = \{a_1, \ldots, a_{d-2},\, b,\, b^{-1}\}$, and $\mathbb{X}_{d,\ell} := (\mathbb{Z}_2 * \cdots * \mathbb{Z}_2)\,(d-2 \text{ folds}\,) * \mathbb{Z}_\ell$ the corresponding Cayley graph; then the transitive graph G is covered by $\mathbb{X}_{d,\ell}$ (see Theorem 11.6 of [33]). From this result, we obtain

$$\lambda_c(G) = \mathrm{gr}(G) \leq \mathrm{gr}(\mathbb{X}_{d,\ell}).$$

For $z \geq 0$, define

$$k_\ell(z) = \begin{cases} 2z + 2z^2 + \cdots + 2z^{\frac{\ell-1}{2}}, & \text{if } \ell \text{ is odd,} \\ 2z + 2z^2 + \cdots + 2z^{\frac{\ell-2}{2}} + z^{\frac{\ell}{2}}, & \text{if } \ell \text{ is even;} \end{cases}$$

$$h_\ell(z) = \frac{(d-2)z}{1+z} + \frac{k_\ell(z)}{1+k_\ell(z)}.$$

Then $\mathrm{gr}(\mathbb{X}_{d,\ell}) = \frac{1}{z_*}$ where z_* is the unique positive number satisfying $h_\ell(z_*) = 1$ (see [9] p. 28). Since $j_\ell := \frac{k_\ell(\frac{1}{d-1})}{1+k_\ell(\frac{1}{d-1})}$ is strictly increasing in ℓ, and $\lim_{r\to\infty} j_r = \frac{2}{d}$, we have $j_\ell < \frac{2}{d}$, which implies $h_\ell(\frac{1}{d-1}) < 1$. Notice that $h_\ell(z)$ is strictly increasing in $z \geq 0$. So $z_* > \frac{1}{d-1}$ and $\mathrm{gr}(\mathbb{X}_{d,\ell}) = \frac{1}{z_*} < d-1$, which implies $\lambda_c(G) < d-1$.

Step 2. Fix $\lambda \in (0,\, d-1)$. Let as before $\mu(o) := d_o$ and $\mu(x) := (d_x^+ + d_x^0 + \lambda d_x^-)\lambda^{-|x|}$ if $x \neq o$. Let $f:\ G \to \mathbb{R}$ be the function defined in (2.3). Then $f \in L^2(G,\, \mu)$.

Since G is transitive, $\lambda_c(G) = \mathrm{gr}(G) = \lim_{n\to\infty} M_n^{1/n}$. By Step 1, for any $\varepsilon \in (0,\, d-1-\lambda_c(G))$, there is a constant $c_\varepsilon > 0$ such that

$$M_n \leq c_\varepsilon\, (\lambda_c(G) + \varepsilon)^n, \qquad \forall n \geq 0.$$

Thus

$$\begin{aligned}\sum_{x\in V(G)} f^2(x)\,\mu(x) &= \sum_{x\in V(G)} \left(1+\frac{d-1-\lambda}{d-1+\lambda}|x|\right)^2 \left(\frac{\lambda}{d-1}\right)^{|x|} \left(d_x^+ + d_x^0 + \lambda d_x^-\right) \lambda^{-|x|} \\ &\le (\lambda\vee 1)d \sum_{n=0}^{\infty} M_n \left(1+\frac{d-1-\lambda}{d-1+\lambda}n\right)^2 \left(\frac{1}{d-1}\right)^n \\ &\le (\lambda\vee 1)dc_\varepsilon \sum_{n=0}^{\infty} \left(\frac{\lambda_c(G)+\varepsilon}{d-1}\right)^n \left(1+\frac{d-1-\lambda}{d-1+\lambda}n\right)^2 \\ &< \infty.\end{aligned}$$

Step 3. (2.6) is true.

Let $\lambda \in (0,\, \lambda_c(\mathbb{T}_d))$. We have noticed in the proof of (i) that $Pf(o) = \rho_{\mathbb{T}_d}(\lambda)f(o)$ and that for $x \neq o$,

$$Pf(x) \ge \rho_{\mathbb{T}_d}(\lambda)f(x), \text{ and `` = '' implies } d_x^- = 1,\, d_x^0 = 0,\, d_x^+ = d-1.$$

Since the transitive G has the minimal cycle length $\ell \ge 3$, we cannot have $d_x^- = 1$, $d_x^0 = 0$, $d_x^+ = d-1$ for any $x \in V(G)\setminus\{o\}$. Note that $f(\cdot)$ and $\mu(\cdot)$ are strictly positive on G. Hence

$$(Pf,\, f) = \sum_{x\in V(G)} Pf(x)f(x)\mu(x) > \sum_{x\in V(G)} \rho_{\mathbb{T}_d}(\lambda)f^2(x)\mu(x) = \rho_{\mathbb{T}_d}(\lambda)(f,\, f).$$

By Step 2, $f \in L^2(G,\, \mu)$, which implies that

$$\rho_G(\lambda) = \sup_{h\in L^2(G,\mu)\setminus\{0\}} \frac{(Ph,\, h)}{(h,\, h)} \ge \frac{(Pf,\, f)}{(f,\, f)} > \rho_{\mathbb{T}_d}(\lambda),$$

proving (2.6). □

Since for some $G \in \mathcal{G}_d$ that are not trees, one may have $\mathrm{gr}(G) = d-1$, in general it is not true that $f \in L^2(G,\, \mu)$ for $\lambda \in (0,\, d-1)$. However, for any transitive graph $G \in \mathcal{G}_d$ that is not isomorphic to $\mathbb{T}_d$, we have $\mathrm{gr}(G) < d-1$, which ensures $f \in L^2(G,\mu)$ in the proof of Theorem 0.2 (ii).

So far we have finished proving Theorem 0.2. □

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